

DEFINABLE TREE PROPERTY CAN HOLD AT ALL UNCOUNTABLE REGULAR CARDINALS

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ABSTRACT. Starting from a supercompact cardinal and a measurable above it, we construct a model of ZFC in which the definable tree property holds at all uncountable regular cardinals. This answers a question from [1]

1. INTRODUCTION

In this paper we study the definable version of Magidor’s question on tree property. Recall that tree property at κ is the assertion “there are no κ -Aronszajn trees”. A question of Magidor asks if it is consistent that tree property holds at all regular cardinals greater than \aleph_1 , and despite of many results which are obtained around it, it seems we are very far from solving it. In this paper we consider the definable version of this question, and give a complete solution to it.

For a regular cardinal κ , let definable tree property at κ , denoted $\text{DTP}(\kappa)$, be the assertion “any κ -tree definable in the structure $(H(\kappa), \in)$ has a cofinal branch”. In [3], it is proved that if κ is regular and $\lambda > \kappa$ is a Π_1^1 -reflecting cardinal, then in the generic extension by the Levy collapse $\text{Col}(\kappa, < \lambda)$, $\text{DTP}(\lambda)$ holds. In [1], this result is extended to get definable tree property at successor of all regular cardinals.

We continue [1] and [3] and prove a global consistency result, by producing a model of ZFC in which definable tree property holds at all uncountable regular cardinals. Our result solves a question of [1] and provides an answer to the definable version of Magidor’s question.

Theorem 1.1. *Assume κ is a supercompact cardinal and $\lambda > \kappa$ is measurable. Then there is a generic extension W of the universe in which the following hold:*

- (a) κ remain inaccessible.
- (b) Definable tree property holds at all uncountable regular cardinals less than κ .

In particular the rank initial segment W_κ of W is a model of ZFC in which definable tree property holds at all uncountable regular cardinals.

Remark 1.2. *We can weaken the large cardinal strength of λ to a Π_1^1 -reflecting cardinal.*

In Section 2, we present some preliminaries and results that will be used in the proof of Theorem 1.1. Then in Section 3, we complete the proof of Theorem 1.1. We assume familiarity with the papers [2] and [4].

2. SOME PRELIMINARIES

We begin with some simple observations and facts that will be used through the paper.

Definition 2.1. *Assume M is a transitive class. Then HMD is the class of all sets which are hereditarily definable using parameters from M .*

In particular, HOD is just HMD, where M is the class of all ordinals. We will use the following well-known fact.

Lemma 2.2. *Assume \mathbb{P} is a homogeneous ordinal definable forcing notion and let G be \mathbb{P} -generic over V . Then $\text{HVD}^{V[G]} \subseteq V$.*

In the next lemma we present a simple proof Leshem's result stated above, using a measurable cardinal.

Lemma 2.3. *Assume κ is regular and $\lambda > \kappa$ is a measurable cardinal. Then $\Vdash_{\text{Col}(\kappa, < \lambda)} \text{"}\lambda = \kappa^+ + \text{DTP}(\lambda)\text{"}$.*

Proof. Let $\mathbb{P} = \text{Col}(\kappa, < \lambda)$ and let G be \mathbb{P} -generic over V . Also let U be a normal measure on λ and let $j : V \rightarrow M \simeq \text{Ult}(V, U)$ be the corresponding ultrapower embedding. By standard arguments, we can lift j to some $j : V[G] \rightarrow M[G \times H]$, where H is $\text{Col}(\kappa, [\lambda, j(\lambda)))$ -generic over $V[G]$ which is defined in $V[G \times H]$.

Now suppose that $T \in V[G]$ is a λ -tree which is definable in $H(\lambda)^{V[G]} = H(\lambda)[G]$. Then in $M[G \times H]$, T has a branch which is defined in a natural way: take some $y \in j(T)_\lambda$ and let

$$b = \{x \in j(T) \mid x <_{j(T)} y\} = \{x \in T \mid x <_{j(T)} y\}.$$

But as the tree is definable and the forcing is homogeneous, we can easily see that b is in fact in $V[G]$: let the formula ϕ and $z \in H(\lambda)^{V[G]}$ be such that

$$\alpha <_T \beta \iff H(\lambda)^{V[G]} \models \phi(z, \alpha, \beta).$$

By the chain condition of forcing, z has a name $\dot{z} \in H(\lambda)^V$, and so $j(\dot{z}) = \dot{z}$. As j is an elementary embedding, we have

$$\alpha <_{j(T)} \beta \iff H(j(\lambda))^{M[G \times H]} \models \phi(z, \alpha, \beta).$$

It follows that $b \in H(M[G])^{M[G \times H]}$, and by the homogeneity of the forcing, $b \in V[G]$. \square

The next lemma is proved in [1].

Lemma 2.4. *Assume $\text{DTP}(\kappa^+)$ holds and let \mathbb{P} be a homogeneous forcing notion which preserves $H(\kappa^+)$. Then $\text{DTP}(\kappa^+)$ holds in the generic extension by \mathbb{P} .*

Proof. Assume G is \mathbb{P} -generic over V and let $T \in V[G]$ be a κ^+ -tree definable in $H(\kappa^+)^{V[G]}$. As $H(\kappa^+)^{V[G]} = H(\kappa^+)^V$, so T is defined by parameters from V and since the forcing is homogeneous, $T \in V$. By our assumption, T has a branch in V and hence in $V[G]$. \square

3. PROOF OF MAIN THEOREM

In this section we complete the proof of Theorem 1.1. Thus assume κ is a supercompact cardinal and let λ be the least measurable cardinal above κ . Let $\bar{E} = \langle E_\xi \mid \xi < \lambda \rangle$ be a Mitchell increasing sequence of extenders such that for each $\xi < \lambda$, $\text{crit}(j_\xi) = \kappa$, ${}^{<\lambda}M \subseteq M$ and $M_\xi \supseteq V_{\kappa+2}$, where $j_\xi : V \rightarrow M_\xi \simeq \text{Ult}(V, E_\xi)$ is the corresponding elementary embedding. Let $\mathbb{P}_{\bar{E}}$ be the supercompact extender based Radin forcing using \bar{E} as defined in [4] and let G be $\mathbb{P}_{\bar{E}}$ -generic over V . Let us recall the basic properties of $\mathbb{P}_{\bar{E}}$.

Lemma 3.1. ([4]) *The following hold in $V[G]$:*

- (a) *There exists a club $C = \langle \kappa_\xi \mid \xi < \kappa \rangle$ of κ consisting of V -measurable cardinals.*
- (b) *For each limit ordinal $\xi < \kappa$ let λ_ξ be the least measurable cardinal above κ_ξ . Then $\lambda_\xi = \kappa_\xi^+$.*
- (c) *κ remains inaccessible.*

Lemma 3.2. *In $V[G]$, definable tree property holds at all λ_ξ , where $\xi < \kappa$ is a limit ordinal.*

Proof. Assume $\xi < \kappa$ is a limit ordinal. Let $p \in G$ be of the form $p = p_0 \widehat{\ } p_1$, where $p_0 \in \mathbb{P}_{\bar{e}}^*$ with $\kappa(\bar{e}) = \kappa_\xi$ and $p_1 \in \mathbb{P}_{\bar{e}}^*$. So we can factor $\mathbb{P}_{\bar{e}}/p$ as $\mathbb{P}_{\bar{e}}/p = \mathbb{P}_{\bar{e}}/p_0 \times \mathbb{P}_{\bar{e}}/p_1$, where $\mathbb{P}_{\bar{e}}/p_0$ is essentially the forcing up to level κ_ξ below p_0 and $\mathbb{P}_{\bar{e}}/p_1$ does not add any new subsets to λ_ξ^+ . Thus it suffices to show that $\text{DTP}(\lambda_\xi)$ holds in the generic extension by $\mathbb{P}_{\bar{e}}/p_0$. This follows by essentially the same ideas as in the proof of Main Theorem 3 from [1] and the homogeneity of the forcing notion $\mathbb{P}_{\bar{e}}/p_0$ as proved in [2].

The forcing $\mathbb{P}_{\bar{e}}/p_0$ is just homogeneous modulo Radin forcing, as we describe it below, hence more work is needed to get the result, and so we sketch the proof for completeness.

Given $p = (f^p, T^p) \in \mathbb{P}_{\bar{e}}^*$, set $s(p) = (f^p \upharpoonright \{\kappa\}, T^p \upharpoonright \{\kappa\})$, and by recursion, define the projection of an arbitrary condition $p = p_0 \widehat{\ } \dots \widehat{\ } p_n \in \mathbb{P}_{\bar{e}}$, by $s(p) = s(p_0 \widehat{\ } \dots \widehat{\ } p_{n-1}) \widehat{\ } s(p_n)$. Set $\mathbb{P}_{\bar{e}}^\pi = s''(\mathbb{P}_{\bar{e}})$. Then $\mathbb{P}_{\bar{e}}^\pi$ is the ordinary Radin forcing using the measure sequence $u = \langle E_\xi(\kappa) \mid \xi < \lambda \rangle$. As shown in [2], the forcing $\mathbb{P}_{\bar{e}}/H$ is homogeneous, where $H = s''(G)$ is $\mathbb{P}_{\bar{e}}^\pi$ -generic over V .

In V , λ_ξ is a measureable cardinal, so let $k : V \rightarrow N$ witness this. It follows that $k(s)$ is a projection from $k(\mathbb{P}_{\bar{e}})$ onto $k(\mathbb{P}_{\bar{e}}^\pi)$.

Now let $T \in V[G]$ be a λ_ξ -tree, which is definable in $H(\lambda_\xi)^{V[G]} = H(\lambda_\xi)^{N[G]}$. We have a natural projection

$$\sigma : k(\mathbb{P}_{\bar{e}}) \rightarrow \mathbb{P}_{\bar{e}}$$

which induces a projection

$$\sigma^\pi : k(\mathbb{P}_{\bar{e}}^\pi) \rightarrow \mathbb{P}_{\bar{e}}^\pi$$

so that the following diagram is commutative

$$\begin{array}{ccc} k(\mathbb{P}_{\bar{e}}) & \xrightarrow{\sigma} & \mathbb{P}_{\bar{e}} \\ k(s) \downarrow & & \downarrow s \\ k(\mathbb{P}_{\bar{e}}^\pi) & \xrightarrow{\sigma^\pi} & \mathbb{P}_{\bar{e}}^\pi \end{array}$$

i.e., $\sigma^\pi \circ k(s) = s \circ \sigma$. Let K be $k(\mathbb{P}_{\bar{e}})$ -generic over V such that $\sigma''(K) = G$. Then $L = k(s)''(K)$ is $k(\mathbb{P}_{\bar{e}}^\pi)$ -generic over V and $\sigma^\pi''(L) = H$.

It is easily seen that we can lift k to some elementary embedding $k : V[G] \rightarrow N[K]$, which is defined in $V[K]$. As in the proof of Lemma 2.3, T has a branch $b \in N[K]$ of the form

$$b = \{x \in k(T) \mid x <_{k(T)} y\} = \{x \in T \mid x <_{k(T)} y\},$$

where $y \in k(T)_{\lambda_\xi}$ is a node on λ_ξ -th level of $k(T)$. Assume ϕ defines T , so that for some $z \in H(\lambda_\xi)^{V[G]}$

$$\alpha <_T \beta \iff H(\lambda_\xi)^{V[G]} \models \phi(z, \alpha, \beta).$$

But $H(\lambda_\xi)^{V[G]} = H(\lambda_\xi)[G]$, so for some $\dot{z} \in H(\lambda_\xi)$, $z = \dot{z}[G]$. We can assume from the start that each element of $H(\lambda_\xi)$ is definable in $H(\lambda_\xi)$ using ordinal parameters less than λ_ξ , so without loss of generality, assume z is an ordinal less than λ_ξ . It follows that $k(z) = z$, and

$$b = \{x \in T \mid H(k(\lambda_\xi))^{N[K]} \models \phi(z, x, y)\} \in \text{HOD}^{N[K]}.$$

Note that k fixes elements of $H(\lambda_\xi)$ and \mathbb{P}_E^π is essentially the Radin forcing at $\kappa_\xi < \lambda_\xi$ using the measure sequence u , so we can easily show that $k(\mathbb{P}_E^\pi)/H$ is homogeneous. On the other hand, using the elementarity of k and by [2], $k(\mathbb{P}_E)/L$ is also homogeneous. It easily follows that $k(\mathbb{P}_E)/H$ is indeed homogeneous, so $\text{HOD}^{N[K]} \subseteq N[H]$ and hence

$$\text{HOD}^{N[K]} \subseteq N[H] \subseteq V[G].$$

It follows that $b \in V[G]$, as required. \square

From now on, assume that $\kappa_0 = \aleph_0$ and that each limit point of C is a singular cardinal in $V[G]$.

Now work in $V[G]$, and let

$$\mathbb{Q} = \langle \langle \mathbb{Q}_\xi \mid \xi \leq \kappa \rangle, \langle \dot{\mathbb{R}}_\xi \mid \xi < \kappa \rangle \rangle$$

be the reverse Easton iteration of forcing notions where for each $\xi < \kappa$, $\Vdash_{\mathbb{Q}_\xi} \text{“}\dot{\mathbb{R}}_\xi = \dot{\text{Col}}(\kappa_\xi^*, < \kappa_{\xi+1})\text{”}$, where $\kappa_\xi^* = \kappa_\xi^+$ if $\xi > 0$ is a limit ordinal and $\kappa_\xi^* = \kappa_\xi$ otherwise. Let H be \mathbb{Q} -generic over $V[G]$.

By Lemmas 2.3 and 2.4,

$$V[G][H] \models \text{“DTP}(\nu^+) \text{ holds for all regular cardinals } \nu < \kappa\text{”}.$$

Note that in $V[G][H]$, there are no inaccessible cardinals below κ (by our assumption on C) and limit cardinals of $V[G][H]$ below κ are of the form κ_ξ , for some limit ordinal $\xi < \lambda$. By 3.1, $(\kappa_\xi^+)^{V[G][H]} = \lambda_\xi$, so the following lemma completes the proof.

Lemma 3.3. $V[G][H] \models \text{“DTP}(\lambda_\xi) \text{ holds for all limit ordinals } \xi < \kappa\text{”}.$

Proof. Let $\mathbb{Q} = \mathbb{Q}_\xi * \dot{\mathbb{Q}}_\infty$, where \mathbb{Q}_ξ is the iteration up to level ξ and $\Vdash_{\mathbb{Q}_\xi} \dot{\mathbb{Q}}_\infty$ is λ_ξ -closed and homogeneous”. So by 2.4, it suffices to show that $\text{DTP}(\lambda_\xi)$ holds in the extension by $\mathbb{P}_{\bar{E}} * \dot{\mathbb{Q}}_\xi$. As before, let $p \in G$ be of the form $p = p_0 \widehat{} p_1$, where $p_0 \in \mathbb{P}_{\bar{e}}^*$ with $\kappa(\bar{e}) = \kappa_\xi$ and $p_1 \in \mathbb{P}_{\bar{E}}^*$ and factor $\mathbb{P}_{\bar{E}}/p$ as $\mathbb{P}_{\bar{E}}/p = \mathbb{P}_{\bar{e}}/p_0 \times \mathbb{P}_{\bar{E}}/p_1$.

As $\mathbb{P}_{\bar{E}}/p_1$ does not add new subsets to λ_ξ^+ , the forcing notion \mathbb{Q}_ξ is computed in both models $V^{\mathbb{P}_{\bar{E}}/p}$ and $V^{\mathbb{P}_{\bar{e}}/p_0}$ in the same way, so it suffices to prove the following:

$$V^{\mathbb{P}_{\bar{e}}/p_0 * \dot{\mathbb{Q}}_\xi} \models \text{“ DTP}(\lambda_\xi) \text{ holds”} .$$

The proof is essentially the same as before using the homogeneity of the corresponding forcing notions . □

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